

Mathematical forms and numerical schemes for solution of unsaturated flow equation

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Abstract.

Prediction of water infiltration into the soil, fluid movement in the unsaturated soils and groundwater recharge are important problems in different fields of science and engineering. Moreover the transfer of the different pollutant (e.g. pesticides) from ground surface to groundwater occurs through the unsaturated zone. In this paper, a comprehensive study on the evaluation of different finite difference schemes (e.g. fully implicit, Crank-Nicolson and Runge-Kutta schemes) for solution of head based and mixed forms of Richards equation is presented. Two examples of water infiltration in very dry and relatively wet unsaturated soils are used for the evaluation of schemes. In addition, the effects of various approximations of moisture capacity function, convergence criteria and time stepping methods on the performance of the schemes are investigated and the results showed their significant influence on mass balance, number of iterations and convergence condition of the numerical schemes. The results of numerical simulations showed that generally mixed form has better performance than head based form. In addition, Crank-Nicolson scheme showed better results than Runge-Kutta scheme, but both of them have convergence problem. In general, the comparison of numerical methods showed that the fully implicit scheme has the best performance among various finite difference schemes and can be selected as a reliable scheme with acceptable solution for different infiltration problems in unsaturated soils.

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1. Introduction

Movement of water within the unsaturated zone is studied within soil physics and hydrology, particularly hydrogeology, and is of importance to agriculture, contaminant transport, and flood control. Groundwater recharge, which is an important process that refills aquifers, generally occurs through the unsaturated zone from precipitation. The Richards equation is often used to mathematically describe the flow of water, which is based partially on Darcy's law. The vertical water flow in unsaturated soil is simulated by combination of Darcy's law and mass conservation equation, yielding the Richards equation (Richards 1931) that can be expressed as different forms. Three standard forms of Richards equation are head based (h-based), water content based (θ -based) and mixed based forms, which are shown as equation (1) (Celia et al. 1990).

$$\begin{aligned} \text{h - Based : } & C(h) \frac{\partial h}{\partial t} - \frac{\partial}{\partial z} \left[k \frac{\partial h}{\partial z} \right] + \frac{\partial k}{\partial z} = 0 \\ \theta \text{ - Based : } & \frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z} \left[D(\theta) \cdot \frac{\partial \theta}{\partial z} \right] + \frac{\partial k}{\partial z} = 0 \\ \text{Mixed Based : } & \frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z} \left[k \frac{\partial h}{\partial z} \right] + \frac{\partial k}{\partial z} = 0 \end{aligned} \quad (1)$$

Where h is pressure head [L], θ is moisture content [$\frac{L^3}{L^3}$], $C(h) = \frac{d\theta}{dh}$ is specific moisture capacity [$\frac{1}{L}$], $k(h)$ is unsaturated hydraulic conductivity [$\frac{L}{T}$], $D(\theta) = \frac{k(\theta)}{C(\theta)}$ is the unsaturated diffusivity [$\frac{L^2}{T}$] and z is the vertical dimension [L] (assumed positive downward).

The Richards equation is a non-linear partial differential equation and there is no analytical solution for it, hence it should be solved numerically.

Among the three forms of Richards equation, the θ -based form is not applicable for saturated soils and it is not easy to simulate the flow in layered soils using θ -based form. The h -based form is more convenient to use than the θ -based form for practical problems such as flow in layered and saturated soils (Huang et al. 1996). Although, implicit and explicit finite difference (FD) schemes for solution of h -based Richards equation can lead to relatively good results (Haverkamp et al. 1977; Huang et al. 1996), further studies showed high mass balance error in the results (van Genuchten 1982; Allen and Murphy 1985; Celia et al. 1990; Hills et al. 1989). Hence, the researchers tried to find numerical algorithms for h -based Richards equation with high accuracy and low mass balance error and computational time (Milly, 1985; Rathfelder and Abriola, 1994; Romano et al. 1999).

Results of numerical solution of mixed based Richards equation showed appropriate mass balance error (Celia et al. 1987; 1990; Zarba, 1988). Celia et al. (1990) used of fully implicit method for solution of Richards equation and showed that mixed based form has better mass

balance than the head based form. In addition, the comparison between numerical solutions of different forms of Richards equation using different FD schemes showed that mixed form has better results than h-based form (Ray and Mohanty, 1992).

The objective of this work is to provide a comprehensive study on the solution of various forms of Richards equation by different FD schemes with considering different numerical parameters. The performance of various implicit FD schemes for numerical solution of h-based and mixed based forms of Richards equation are investigated. Also, the effects of using of various convergence criteria (Huang et al., 1996), time stepping methods and approximations of specific moisture capacity (Rathfelder and Abriola, 1994) on the results of different schemes are evaluated. Finally, the appropriate and reliable methods for solution of Richards equation are determined.

2. Materials and methods

2.1. Finite difference approximations

The discussed FD schemes in this study are consisted of fully implicit, Crank-Nicolson (C-N) and Runge-Kutta (One step-two stages method). The discretization form of each scheme has been presented in the Appendixes I and II for h-based and mixed forms of Richards equation, respectively. The general discretized form of Richards equation after discretization, linearization and simplification can be expressed as equation (2).

$$Ah_{i-1}^{n+1,m+1} + Bh_i^{n+1,m+1} + Dh_{i+1}^{n+1,m+1} = E \quad (2)$$

Where, $h_i^{n+1,m+1}$ is the pressure head value in node i at time level $n+1$ and iteration stage $m+1$. The coefficients A , B , D and E have different expressions in the different numerical schemes. Tables 1 and 2 show the expression of these coefficients for discretized h-based and mixed forms of Richards equation by different FD schemes, respectively. In these tables, the Δz is constant spacing, Δt is the time step, $k_i^{n+1,m}$, $C_i^{n+1,m}$ and $\theta_i^{n+1,m}$ are the hydraulic conductivity, moisture capacity and moisture content in node i at time level $n+1$ and in iteration stage m . h_i^n is the pressure head in node i and at time level n . k_i^* and C_i^* are $k(h_i^*)$ and $C(h_i^*)$, respectively which h_i^* can be calculated by equation (3):

$$h_i^* = \frac{\Delta t}{4C_i^n} \left[\frac{k_{i+1}^n + k_i^n}{2} \times \frac{(h_{i+1}^n - h_i^n)}{\Delta z^2} - \frac{k_i^n + k_{i-1}^n}{2} \times \frac{(h_i^n - h_{i-1}^n)}{\Delta z^2} - \frac{k_{i+1}^n - k_{i-1}^n}{2\Delta z} \right] + h_i^n \quad (3)$$

Where, k_i^n , C_i^n and h_i^n are the hydraulic conductivity, specific moisture capacity and pressure head in node i and at time level n .

The iteration method for solution of mentioned discretized h-based and mixed forms of Richards equation are Picard and modified Picard iteration methods, respectively. For details about these iteration methods refer to Celia et al. (1990) and Zarba (1988).

For implementation of each numerical method, some parameters, criteria and estimations should be defined. These are introduced in the next sections.

2.2. Approximations of specific moisture capacity

The evaluation of C (specific moisture capacity function) can be performed with two different methods. The first method is the evaluation of C with analytical derivatives of $\theta(h)$ that is named tangent approximation ($C(h) = \frac{d\theta}{dh}$) and the second method is standard chord slope (SCS) approximation (Rathfelder and Abriola 1994). The SCS method is expressed by equation (4):

$$C_i^{n+1,m} = \frac{\theta_i^{n+1,m} - \theta_i^n}{h_i^{n+1,m} - h_i^n} \quad (4)$$

In spite of tangent method, using of SCS method can improve the mass balance of numerical scheme for h-based Richards equation (Rathfelder and Abriola 1994). In this study, the tangent and SCS approximation are examined with all of different FD schemes.

2.3. Convergence Criteria

In the iterative methods (e.g. Picard method), the iterative process continues unless the difference of calculated pressure heads between two successive iteration levels in each node becomes less than a predefined tolerance (δa) as shown in equation (5). This criterion is named Standard convergence criterion.

$$\left| h^{n+1,m+1} - h^{n+1,m} \right| \leq \delta a \quad (5)$$

Although it is true that more accurate solution can be obtained with smaller values of tolerance, but the computational time increases. δa is widely varying, for instance $\delta a = 0.01$ cm adopted for simulation three dimensional flow in unsaturated soil using finite element scheme but $\delta a = 1$ cm utilized for drainage simulation (Huyakorn and Wadsworth 1985). In this study, $\delta a = 0.1$ cm has been adopted for numerical solutions.

The second suggested convergence criterion is the use of an empirical convergence criterion, named mixed convergence criterion and it involves both absolute (δa) and relative error (δr) as presented in equation (6) (Kaluarachchi and Parker 1989; Cooley 1983).

$$\left| h^{n+1,m+1} - h^{n+1,m} \right| \leq \delta r \left| h^{n+1,m+1} \right| + \delta a \quad (6)$$

The adopted values for the relative tolerance (δr) have ranged from 0.001 to 0.01, depending upon the desired accuracy. In this study, $\delta r = 0.001$ has been adopted.

The third convergence criterion is named nonlinear convergence criterion and it is based on the difference between the water contents values in two successive iterations; therefore this criterion should be expressed as equation (7).

$$C^{n+1,m} \left| h_i^{n+1,m+1} - h_i^{n+1,m} \right| = \left| \theta^{n+1,m+1} - \theta^{n+1,m} \right| \leq \delta_\theta \quad (7)$$

δ_θ has no a distinct value. Huang et al. (1996) used $\delta_\theta = 0.0001$, hence δ_θ is considered equal to 0.0001 in this study.

In this study, the effect of different convergence criteria on the numerical schemes is evaluated.

2.4. Time stepping methods

If the time step be constant and predefined, it is named constant time stepping. Another approach is automatic time stepping. Equation (8) presents the automatic time stepping method.

$$\text{If } N \leq 3 \text{ then } \Delta t^{\text{new}} = \mu_1 \Delta t^{\text{old}}$$

$$\text{If } 3 < N \leq 7 \text{ then } \Delta t^{\text{new}} = \Delta t^{\text{old}}$$

$$\text{If } 7 < N \leq N_{\text{max}} \text{ then } \Delta t^{\text{new}} = \mu_2 \Delta t^{\text{old}}$$

$$\text{If } N_{\text{max}} < N \text{ then } \Delta t^{\text{new}} = \frac{\Delta t^{\text{old}}}{3}$$

(8)

Which, N is the iteration number in each stage, N_{max} is the maximum number of iterations that

its range is from 10 to 50. Δt^{old} and Δt^{new} are the old and new time step values, respectively. μ_1 and μ_2 are the constant coefficients and their typical ranges are from 1.1 to 1.5 and 0.3 to 0.9, respectively.

Equation (8) shows that if the number of iterations is small, the time step can be increased. In addition, if the number of iterations is more than maximum number of iterations, it shows that smaller time step is required. In this study, the values of N_{max} , μ_1 and μ_2 are considered to be 30, 1.5 and 0.7, respectively.

In addition, performance of the constant and automatic time stepping methods are evaluated in this research.

2.5. Evaluation criteria

Three criteria are considered for evaluation of the performance of numerical methods and their comparisons. These criteria are Root Mean Square Error (RMSE), mass balance conservation and number of iterations.

Magnuson et al. (1990) and El-kadi and Ling (1993) used RMSE for evaluation of numerical methods.

Another evaluation method is the ability to conserve global mass over the domain of interest (mass balance conservation). Although adequate mass conservation is necessary but it is not sufficient for acceptability of a numerical scheme (Celia et al. 1990). Mass balance (MB) can be defined as equation (9) (Celia et al. 1990):

$$MB(t) = \frac{\text{total additional mass in the domain}}{\text{total net flux into the domain}} \quad (9)$$

The added water in the system in the time t is calculated according to the system initial mass. For FD approximation with the first type boundary conditions, MB is calculated as equation (10) (Celia et al. 1990):

$$MB(t^n) = \frac{\sum_{i=2}^{N-1} (\theta_i^n - \theta_i^0) \Delta z}{\sum_{j=0}^n \left\{ -k_{N-1/2}^j \left[\frac{h_{N-1}^j - h_N^j}{\Delta z} + 1 \right] + k_{3/2}^j \left[\frac{h_1^j - h_2^j}{\Delta z} + 1 \right] \right\} \Delta t} \quad (10)$$

Where, $k_{3/2}^j = \frac{k_1^j + k_2^j}{2}$, $k_{N-1/2}^j = \frac{k_N^j + k_{N-1}^j}{2}$, N is the number of nodes, n is the number of time steps, h_N^j is the pressure head in the j -th time step and N -th node, θ_i^0 and θ_i^n are the initial and final values of moisture content in node i , respectively.

2.6. Numerical examples

Two one-dimensional infiltration examples are investigated in this study.

Haverkamp et al. (1977) experiment was considered as the first numerical example. It was a series of laboratory infiltration experiments using a plexiglass column with 6 cm inside diameter,

packed by sand. The bulk density of soil column was 1.66 g/cm^3 . The water content in the soil column was measured using gamma ray attenuation method. In addition, some tensiometers were installed in different depths of column to measure the water pressure head. The initial water content of column was $0.1 \text{ cm}^3/\text{cm}^3$ (or pressure head equal to -61.5 cm). A constant water pressure head ($h = -61.5 \text{ cm}$) was maintained in the lower end of column. Surface water content was constant and equal to $0.267 \text{ cm}^3/\text{cm}^3$ (or pressure head equal to -20.7 cm). The following constitutive equations derived by a least square fit on the all of the experimental data (Haverkamp et al. 1977):

$$\theta(h) = \frac{\alpha(\theta_s - \theta_r)}{\alpha + |h|^\beta} + \theta_r \quad (11)$$

$$k(h) = k_s \frac{\rho}{\rho + |h|^\gamma} \quad (12)$$

Where, α, β, ρ and γ are fitting parameters with the values of 1.611×10^6 , 3.96 , 1.175×10^6 and 4.74 , respectively. The saturated and residual water content (θ_s and θ_r) are 0.287 and 0.075 , respectively. The saturated hydraulic conductivity of the soil (k_s) is 0.00944 cm/s .

The second example is a vertical column of very dry soil (100 cm), presented by Celia et al (1990). This numerical example has been used by different researchers such as Celia et al. (1990), El-kadi and Ling (1993) and Rathfelder and Abriola (1994). This numerical example is

an approximate description of a field site in New Mexico (Celia et al. 1990). The initial condition was set as a constant pressure head of -10 m, while the upper and lower boundary values were set at -75 cm ($\theta=0.2003$) and -1000 cm ($\theta=0.1099$), respectively. The saturated hydraulic conductivity is 0.00922 cm/s, and the saturated and residual water contents are 0.368 and 0.102, respectively. Van Genuchten equations are used as constitutive relationships of this soil (Van Genuchten 1980):

$$\theta(h) = \frac{\theta_s - \theta_r}{\left[1 + (\alpha' |h|)^n\right]^{m'}} + \theta_r \quad (13)$$

$$k(h) = k_s \frac{\left\{1 - (\alpha' |h|)^{n'-1} \left[1 + (\alpha' |h|)^n\right]^{-m'}\right\}^2}{\left[1 + (\alpha' |h|)^n\right]^{m'/2}} \quad (14)$$

Which m' , n' and α' are fitting parameters and their values are 0.5, 2.0 and 0.0335, respectively.

The examples 1 and 2 are infiltration into the initially relatively wet and very dry soils, respectively.

3. Results and Discussions

In this study, all of the numerical codes have been written in MATLAB 6.0 software.

The example 1 was solved by different numerical schemes as 40 cm column with vertical grid

spacing of $\Delta z = 1$ cm and elapsed time of 360 s. The example 2 was solved by different numerical schemes as 1 m column of soil with vertical grid spacing of $\Delta z = 2.5$ cm and elapsed time of 86400 s.

Numerical solutions of two examples with very dense grid in space and time were considered as the exact solutions for determination of RMSE values of numerical schemes.

3.1. Crank-Nicolson scheme:

The results of C-N scheme with tangent approximation of storage term have been presented in tables 3 and 4. These results show that the C-N scheme does not converge appropriately.

Similar to tangent approximation, the Crank-Nicolson scheme with SCS approximation and equation (7) as convergence criterion didn't converge at the any Δt conditions, presented in tables 3 and 4.

The example 1 with constant time stepping converges appropriately in the time step (Δt) 1.6 and 20 s for h-based and mixed forms, respectively. The example 2 with constant time stepping converges appropriately in the time step values 5 and 75 s in h-based and mixed forms, respectively. The results demonstrated that the C-N scheme needs to small time step values for convergence and mixed form solution converges a little better than h-based form.

In addition, the results imply that the automatic time stepping method can improve the convergence condition and the non-linear convergence criterion is the main factor of decreasing the number of iterations. These results are similar to the findings of Huang et al. (1996).

According to the tables 3 and 4, when C-N scheme converges, it can lead to appropriate numerical results, but because of convergence problem of C-N scheme, it cannot be adopted as a reliable scheme for the solution of Richards equation.

3.2. Runge-Kutta scheme

The results of numerical solution by Runge-Kutta scheme showed worse results than C-N scheme. Runge-Kutta scheme could not converge in $\Delta t = 10$ and $\Delta t = 20$ sec in examples 1 and 2, respectively. It needs to very small Δt values for convergence. The results of numerical solutions using Runge-Kutta scheme showed that mixed form has no preference to h-based form.

3.3. Fully implicit scheme

Tables 5 and 6 show the results of fully implicit scheme using tangent approximation of specific moisture capacity for examples 1 and 2.

The results show that in spite of the poor mass balance of h-based form, the mixed form formulation has accurate mass balance. Similar findings have been reported by Celia et al. (1990).

Table 6 ($\Delta t = 144$ s) shows that the mixed form has better convergence than h-based form. According to the results of fully implicit scheme (see tables 5 and 6), the mixed form has better performance than h-based form. Similar to the results of C-N scheme, automatic time stepping method improved the convergence condition of the scheme and non-linear convergence criterion

significantly decreased its number of iterations.

Figure 1 exhibits the results of h-based and mixed forms using tangent approximation for example 1. This figure implies that the error increases as the time step increases but the solutions are still convergent for two studied forms of Richards equation. The h-based form underestimates the infiltration front and as the time step increases, the error in front increases. In addition, it has overestimated the infiltration depth. For example, it shows more than 6% and 20% error in infiltration depth for $\Delta t=30$ s and $\Delta t=120$ s, respectively. The mixed form has presented better results for infiltration front and in spite of h-based form, it overestimates the infiltration front. Similar to the h-based form, it overestimate the infiltration depth and its errors for $\Delta t=30$ s and $\Delta t=120$ s are more than 9% and 25%, respectively.

Figures 2 exhibits the results of h-based and mixed forms using tangent approximation for example 2. For h-based form, when the time step is small ($\Delta t=30$ s), then the infiltration front and infiltration depth are overestimated and as Δt increases ($\Delta t=3600$ s), the infiltration front and depth are underestimated. In addition, the results of mixed form for $\Delta t=20$, 144 and 720 s have been overlapped, hence the mixed form implies that its result is almost independent to time step. Overestimation of infiltration depth for $\Delta t=20$, 144 and 720 s and $\Delta t=3600$ s are about 8% and 11%, respectively.

In general, the mentioned comparisons between h-based and mixed form demonstrated that mixed form is better and more reliable method than h-based form.

Rathfelder and Abriola (1994) found that replacement of SCS approximation with tangent

approximation in the numerical scheme can remove the mass balance problem of h-based form. Hence, we used of SCS approximation instead of tangent approximation. The results of numerical schemes after this replacement and utilizing constant time stepping method have been presented in tables 7 and 8. As you see in these tables, there is no significant difference between the results of h-based and mixed forms. Comparison among tables 5-8 demonstrated that, although, the replacement of tangent approximation with SCS approximation has no significant effect on the solution results of mixed form but it can completely remove the mass balance weaknesses of h-based form. In addition, comparison between tables 6 and 8 ($\Delta t = 144$ s) shows that numerical scheme with SCS approximation has better convergence than tangent approximation. The results certify the findings of Rathfelder and Abriola (1994) and the mass balance of h-based form has improved. Figures 3 and 4 show the results of h-based form with SCS approximation and constant time stepping method for examples 1 and 2, respectively. Similar to figure 2b, the numerical results for $\Delta t=20, 144$ and 720 s have been overlapped in figure 4. The figures exhibit the similarity of h-based form with SCS approximation results with the results of mixed form.

In general, comparison between different schemes (Crank-Nicolson, Runge-Kutta and fully implicit schemes) shows that fully implicit scheme is the best one and it can be selected as an appropriate and reliable method for solution of water flow equation and estimation of water flow in different unsaturated soils. Using SCS approximation for specific moisture capacity instead of tangent approximation can lead to the better mass balance and convergence.

4. Conclusions

The unsaturated zone is a fundamental part of the hydrologic cycle. It essentially controls interrelationships between precipitation, infiltration, surface runoff, evapotranspiration and groundwater recharge. The transfer of water from the land surface to groundwater and vice versa is regulated by unsaturated zone. Also it has a main role in providing protection, screening, filtering, transfer and attenuation of potential groundwater contaminants that are delivered via the land surface. Accurate and robust solution of the unsaturated flow equation is an imperative part of any practical study in the field of water resources. A comprehensive study of various finite difference schemes (e.g. fully implicit, Crank-Nicolson and Runge-Kutta schemes) for solution of head based and mixed forms of Richards equation performed in this paper.

The Runge-Kutta (one step two stages method) and C-N scheme encounter convergence problem and they are not suggested for numerical solution of Richards equation.

The automatic time stepping method can improve the convergence condition of the scheme. In addition, non-linear convergence criterion can significantly decrease the number of iterations.

Fully implicit scheme with tangent approximation of specific moisture capacity, h-based and mixed forms showed poor and excellent mass balances, respectively. In addition, mixed form has a better convergence than h-based form. Generally, mixed form is superior and more reliable than h-based form for solution of Richards equation.

Replacement of tangent approximation with SCS approximation not only removed the mass balance problem of h-based form but also improved its convergence condition.

Numerical solution of h-based and mixed based forms by fully implicit scheme and SCS approximation show similar results.

Fully implicit scheme enhanced the results in comparison with C-N and Runge-Kutta schemes and it can be considered as the appropriate and reliable scheme.

In general, for numerical modeling of unsaturated flow equation, the fully implicit scheme with SCS approximation for specific moisture capacity can be recommended when the h-based form of Richards equation is solved.

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Appendix I

Different FD discretization for h-based form

The following discretization forms can be converted to the equation 4 and the presented coefficients in table 1 (*A*, *B*, *D* and *E*) can be calculated easily.

Fully implicit scheme:

$$C \frac{\partial h}{\partial t} - \frac{\partial \left[k \frac{\partial h}{\partial z} \right]}{\partial z} + \frac{\partial k}{\partial z} = 0 \quad (15)$$

$$C_i^{n+1,m} \times \frac{h_i^{n+1,m+1} - h_i^n}{\Delta t} - \frac{1}{\Delta z} \left[\frac{k_{i+1/2}^{n+1,m} (h_{i+1}^{n+1,m+1} - h_i^{n+1,m+1}) - k_{i-1/2}^{n+1,m} (h_i^{n+1,m+1} - h_{i-1}^{n+1,m+1})}{\Delta z} \right] + \frac{k_{i+1/2}^{n+1,m} - k_{i-1/2}^{n+1,m}}{\Delta z} = 0 \quad (16)$$

Crank-Nicolson scheme:

$$C \frac{\partial h}{\partial t} - \left[\frac{\partial k}{\partial z} \times \frac{\partial h}{\partial z} + k \frac{\partial^2 h}{\partial z^2} \right] + \frac{\partial k}{\partial z} = 0 \quad (17)$$

$$C_i^{n+1,m} \times \frac{h_i^{n+1,m+1} - h_i^n}{\Delta t} - \left[\frac{(k_{i+1/2}^{n+1,m} - k_{i-1/2}^{n+1,m})}{4\Delta z} \times \frac{(h_{i+1}^{n+1,m+1} - h_{i-1}^{n+1,m+1} + h_{i+1}^n - h_{i-1}^n)}{\Delta z} + \right. \\ \left. \frac{(k_{i+1/2}^{n+1,m} + k_{i-1/2}^{n+1,m})}{2} \times \left(\frac{h_{i+1}^{n+1,m+1} - 2h_i^{n+1,m+1} + h_{i-1}^{n+1,m+1} + h_{i+1}^n - 2h_i^n + h_{i-1}^n}{2\Delta z^2} \right) \right] + \frac{k_{i+1/2}^{n+1,m} - k_{i-1/2}^{n+1,m}}{\Delta z} = 0 \quad (18)$$

Runge-Kutta scheme:

$$\frac{\partial h}{\partial t} = \frac{1}{C} \left[\frac{\partial(k \frac{\partial h}{\partial z})}{\partial z} - \frac{\partial k}{\partial z} \right] = F(h, z, t) \quad (19)$$

$$\frac{h^* - h^n}{\Delta t} = \frac{F^n}{4} \quad (20)$$

$$\frac{h^* - h^n}{\Delta t} = \frac{1}{4C_i^n} \left[\frac{1}{\Delta z} \left(\frac{k_{i+1/2}^n (h_{i+1}^n - h_i^n) - k_{i-1/2}^n (h_i^n - h_{i-1}^n)}{\Delta z} \right) - \frac{k_{i+1/2}^{n+1,m} - k_{i-1/2}^{n+1,m}}{\Delta z} \right] \quad (21)$$

$$\frac{h^{n+1} - h^n}{\Delta t} = \frac{2}{3} F^* + \frac{1}{3} F^{n+1} \quad (22)$$

$$\frac{h_i^{n+1,m+1} - h_i^n}{\Delta t} = \frac{2}{3C_i^*} \left[\frac{1}{\Delta z} \left(\frac{k_{i+1/2}^* (h_{i+1}^* - h_i^*) - k_{i-1/2}^* (h_i^* - h_{i-1}^*)}{\Delta z} \right) - \frac{k_{i+1/2}^* - k_{i-1/2}^*}{\Delta z} \right] + \\ \frac{1}{3C_i^{n+1,m}} \left[\frac{1}{\Delta z} \left(\frac{k_{i+1/2}^{n+1,m} (h_{i+1}^{n+1,m+1} - h_i^{n+1,m+1}) - k_{i-1/2}^{n+1,m} (h_i^{n+1,m+1} - h_{i-1}^{n+1,m+1})}{\Delta z} \right) - \frac{k_{i+1/2}^{n+1,m} - k_{i-1/2}^{n+1,m}}{\Delta z} \right] \quad (23)$$

Where, $k_{i+1/2}^{n+1,m} = \frac{k_{i+1}^{n+1,m} + k_i^{n+1,m}}{2}$, $k_{i-1/2}^{n+1,m} = \frac{k_i^{n+1,m} + k_{i-1}^{n+1,m}}{2}$

Appendix II

Different FD discretization for mixed form

The following discretization forms can be converted to the equation 4 and the presented coefficients in table 2 (*A*, *B*, *D* and *E*) can be calculated easily.

Fully implicit scheme:

$$\frac{\partial \theta}{\partial t} - \frac{\partial \left[k \frac{\partial h}{\partial z} \right]}{\partial z} + \frac{\partial k}{\partial z} = 0 \quad (24)$$

$$\frac{\theta_i^{n+1,m+1} - \theta_i^n}{\Delta t} - \frac{1}{\Delta z} \left[\frac{k_{i+1/2}^{n+1,m} (h_{i+1}^{n+1,m+1} - h_i^{n+1,m+1}) - k_{i-1/2}^{n+1,m} (h_i^{n+1,m+1} - h_{i-1}^{n+1,m+1})}{\Delta z} \right] + \frac{k_{i+1/2}^{n+1,m} - k_{i-1/2}^{n+1,m}}{\Delta z} = 0 \quad (25)$$

The Taylor expansion of $\theta_i^{n+1,m+1}$ is:

$$\theta_i^{n+1,m} = \theta_i^{n+1,m} + C_i^{n+1,m} (h_i^{n+1,m+1} - h_i^{n+1,m}) + O\delta^2$$

Therefore:

$$\frac{\theta_i^{n+1,m} + C_i^{n+1,m} (h_i^{n+1,m+1} - h_i^{n+1,m}) - \theta_i^n}{\Delta t} - \frac{1}{\Delta z} \left[\frac{k_{i+1/2}^{n+1,m} (h_{i+1}^{n+1,m+1} - h_i^{n+1,m+1}) - k_{i-1/2}^{n+1,m} (h_i^{n+1,m+1} - h_{i-1}^{n+1,m+1})}{\Delta z} \right] + \frac{k_{i+1/2}^{n+1,m} - k_{i-1/2}^{n+1,m}}{\Delta z} = 0 \quad (26)$$

Crank-Nicolson scheme:

According to the fully implicit scheme and h-based form, the discretization form for C-N scheme will be as the following equation:

$$\frac{\theta_i^{n+1,m} + C_i^{n+1,m} (h_i^{n+1,m+1} - h_i^{n+1,m}) - \theta_i^n}{\Delta t} - \left[\frac{(k_{i+1/2}^{n+1,m} - k_{i-1/2}^{n+1,m})}{4\Delta z} \times \frac{(h_{i+1}^{n+1,m+1} - h_{i-1}^{n+1,m+1} + h_{i+1}^n - h_{i-1}^n)}{\Delta z} + \frac{(k_{i+1/2}^{n+1,m} + k_{i-1/2}^{n+1,m})}{2} \times \left(\frac{h_{i+1}^{n+1,m+1} - 2h_i^{n+1,m+1} + h_{i-1}^{n+1,m+1} + h_{i+1}^n - 2h_i^n + h_{i-1}^n)}{2\Delta z^2} \right) \right] + \frac{k_{i+1/2}^{n+1,m} - k_{i-1/2}^{n+1,m}}{\Delta z} = 0 \quad (27)$$

Runge-Kutta scheme:

$$\frac{\partial \theta}{\partial t} = \frac{1}{C} \left[\frac{\partial (k \frac{\partial h}{\partial z})}{\partial z} - \frac{\partial k}{\partial z} \right] = F(h, z, t) \quad (28)$$

$$\frac{\theta^* - \theta^n}{\Delta t} = \frac{F^n}{4} \quad (29)$$

$$\frac{\theta^* - \theta^n}{\Delta t} = \frac{1}{4C_i^n} \left[\frac{1}{\Delta z} \left(\frac{k_{i+1/2}^n (h_{i+1}^n - h_i^n) - k_{i-1/2}^n (h_i^n - h_{i-1}^n)}{\Delta z} \right) - \frac{k_{i+1/2}^{n+1,m} - k_{i-1/2}^{n+1,m}}{\Delta z} \right] \quad (30)$$

$$\frac{\theta^{n+1} - \theta^n}{\Delta t} = \frac{2}{3} F^{*} + \frac{1}{3} F^{n+1} \quad (31)$$

$$\frac{\theta_i^{n+1,m+1} - \theta^n}{\Delta t} = \frac{2}{3C_i^*} \left[\frac{1}{\Delta z} \left(\frac{k_{i+1/2}^* (h_{i+1}^* - h_i^*) - k_{i-1/2}^* (h_i^* - h_{i-1}^*)}{\Delta z} \right) - \frac{k_{i+1/2}^* - k_{i-1/2}^*}{\Delta z} \right] + \frac{1}{3C_i^{n+1,m}} \left[\frac{1}{\Delta z} \left(\frac{k_{i+1/2}^{n+1,m} (h_{i+1}^{n+1,m+1} - h_i^{n+1,m+1}) - k_{i-1/2}^{n+1,m} (h_i^{n+1,m+1} - h_{i-1}^{n+1,m+1})}{\Delta z} \right) - \frac{k_{i+1/2}^{n+1,m} - k_{i-1/2}^{n+1,m}}{\Delta z} \right] \quad (32)$$

According to the fully implicit scheme and h-based form, the final discretization form will be as the following equation:

$$\frac{\theta_i^{n+1,m} + C_i^{n+1,m} (h_i^{n+1,m+1} - h_i^{n+1,m}) - \theta_i^n}{\Delta t} = \frac{2}{3C_i^*} \left[\frac{1}{\Delta z} \left(\frac{k_{i+1/2}^* (h_{i+1}^* - h_i^*) - k_{i-1/2}^* (h_i^* - h_{i-1}^*)}{\Delta z} \right) - \frac{k_{i+1/2}^* - k_{i-1/2}^*}{\Delta z} \right] + \frac{1}{3C_i^{n+1,m}} \left[\frac{1}{\Delta z} \left(\frac{k_{i+1/2}^{n+1,m} (h_{i+1}^{n+1,m+1} - h_i^{n+1,m+1}) - k_{i-1/2}^{n+1,m} (h_i^{n+1,m+1} - h_{i-1}^{n+1,m+1})}{\Delta z} \right) - \frac{k_{i+1/2}^{n+1,m} - k_{i-1/2}^{n+1,m}}{\Delta z} \right] \quad (33)$$

Table 1. *A*, *B*, *D* and *E* coefficients for h-based form of Richards equation

	Fully implicit	Crank-Nicolson	Range-Kutta
A	$-\frac{k_i^{n+1,m} + k_{i-1}^{n+1,m}}{2 \Delta z^2}$	$-\frac{k_i^{n+1,m} + k_{i-1}^{n+1,m}}{4 \Delta z^2}$	$\frac{-k_i^{n+1,m} + k_{i-1}^{n+1,m}}{6 \Delta z^2}$
B	$\frac{C_i^{n+1,m}}{\Delta t} + \frac{k_{i+1}^{n+1,m} + 2k_i^{n+1,m} + k_{i-1}^{n+1,m}}{2 \Delta z^2}$	$\frac{C_i^{n+1,m}}{\Delta t} + \frac{k_{i+1}^{n+1,m} + 2k_i^{n+1,m} + k_{i-1}^{n+1,m}}{4 \Delta z^2}$	$\frac{C_i^{n+1,m}}{\Delta t} + \frac{k_{i+1}^{n+1,m} + 2k_i^{n+1,m} + k_{i-1}^{n+1,m}}{6 \Delta z^2}$
D	$-\frac{k_i^{n+1,m} + k_{i+1}^{n+1,m}}{2 \Delta z^2}$	$-\frac{k_i^{n+1,m} + k_{i+1}^{n+1,m}}{4 \Delta z^2}$	$\frac{-k_i^{n+1,m} + k_{i+1}^{n+1,m}}{6 \Delta z^2}$
E	$\frac{-k_{i+1}^{n+1,m} + k_{i-1}^{n+1,m}}{2 \Delta z} + C_i^{n+1,m} \frac{h_i^n}{\Delta t}$	$\frac{-k_{i+1}^{n+1,m} + k_{i-1}^{n+1,m}}{2 \Delta z} + C_i^{n+1,m} \frac{h_i^n}{\Delta t} + \left[\frac{k_{i+1}^{n+1,m} + k_i^{n+1,m}}{2} \times (h_{i+1}^n - h_i^n) - \frac{k_{i-1}^{n+1,m} + k_i^{n+1,m}}{2} \times (h_i^n - h_{i-1}^n) \right] / (2 \Delta z^2)$	$\frac{-k_{i+1}^{n+1,m} + k_{i-1}^{n+1,m}}{6 \Delta z} + \frac{C_i^{n+1,m} \times h_i^n}{\Delta t} + \frac{2C_i^{n+1,m}}{3C_i^*} \times \left[\frac{(k_{i+1}^* + k_i^*)(h_{i+1}^* - h_i^*)}{2 \Delta z^2} - \frac{(k_{i-1}^* + k_i^*)(h_i^* - h_{i-1}^*)}{2 \Delta z^2} \right] + \frac{(-k_{i+1}^* + k_{i-1}^*)}{2 \Delta z} J$

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Table 2. *A*, *B*, *D* and *E* coefficients for mixed form of Richards equation

	Fully implicit	Crank-Nicolson	Range-Kutta
A	$-\frac{k_i^{n+1,m} + k_{i-1}^{n+1,m}}{2 \Delta z^2}$	$-\frac{k_i^{n+1,m} + k_{i-1}^{n+1,m}}{4 \Delta z^2}$	$\frac{(-k_i^{n+1,m} + k_{i-1}^{n+1,m})}{6 \Delta z^2}$
B	$\frac{C_i^{n+1,m}}{\Delta t} + \frac{k_{i+1}^{n+1,m} + 2k_i^{n+1,m} + k_{i-1}^{n+1,m}}{2 \Delta z^2}$	$\frac{C_i^{n+1,m}}{\Delta t} + \frac{k_{i+1}^{n+1,m} + 2k_i^{n+1,m} + k_{i-1}^{n+1,m}}{4 \Delta z^2}$	$\frac{C_i^{n+1,m}}{\Delta t} + \frac{k_{i+1}^{n+1,m} + 2k_i^{n+1,m} + k_{i-1}^{n+1,m}}{6 \Delta z^2}$
D	$-\frac{k_i^{n+1,m} + k_{i+1}^{n+1,m}}{2 \Delta z^2}$	$-\frac{k_i^{n+1,m} + k_{i+1}^{n+1,m}}{4 \Delta z^2}$	$\frac{(-k_i^{n+1,m} + k_{i+1}^{n+1,m})}{6 \Delta z^2}$
E	$C_i^{n+1,m} \frac{h_i^{n+1,m}}{\Delta t} - \frac{\theta_i^{n+1,m} - \theta_i^n}{\Delta t} + \frac{-k_{i+1}^{n+1,m} + k_{i-1}^{n+1,m}}{2 \Delta z}$	$C_i^{n+1,m} \frac{h_i^{n+1,m}}{\Delta t} - \frac{\theta_i^{n+1,m} - \theta_i^n}{\Delta t} + \left[\frac{k_{i+1}^{n+1,m} + k_i^{n+1,m}}{2} (h_{i+1}^n - h_i^n) - \frac{k_{i-1}^{n+1,m} + k_i^{n+1,m}}{2} \times (h_i^n - h_{i-1}^n) \right] / (2 \Delta z^2)$	$\frac{-k_{i+1}^{n+1,m} + k_{i-1}^{n+1,m}}{6 \Delta z} + C_i^{n+1,m} \frac{h_i^{n+1,m}}{\Delta t} - \frac{\theta_i^{n+1,m} - \theta_i^n}{\Delta t} + \frac{2}{3} \times \left[\frac{(k_{i+1}^* + k_i^*)(h_{i+1}^* - h_i^*)}{2 \Delta z^2} - \frac{(k_{i-1}^* + k_i^*)(h_i^* - h_{i-1}^*)}{2 \Delta z^2} + \frac{-k_{i+1}^* + k_{i-1}^*}{2 \Delta z} \right]$

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Table 3. Results of numerical solution of example 1 using C-N scheme with tangent approximation ($t=360$ s and $\Delta z = 1$ cm)

			MB		RMSE (cm)		Iteration Numbers	
Time Stepping		Convergence	Δt (s)		Δt (s)		Δt (s)	
Method		criterion	10	30	10	30	10	30
h- based	Constant	Eq. (5)	-	-	-	-	-	-
	Automatic	Eq. (6)	0.99	0.99	0.04	0.04	586	587
		Eq. (7)	0.97	0.97	0.1	0.1	143	144
Mixed	Constant	Eq. (5)	29.8	-	0.52	-	230	-
	Automatic	Eq. (6)	4.4	1.01	0.57	0.14	389	581
		Eq. (7)	5.4	-	0.43	-	94	-

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Table 4. Results of numerical solution of example 2 using C-N scheme with tangent approximation ($t=86400$ s and $\Delta z = 2.5$ cm)

			MB		RMSE (cm)		Iteration Numbers	
Time Stepping Method		Convergence criterion	Δt (s)		Δt (s)		Δt (s)	
			20	144	20	144	20	144
h-based	Constant	Eq. (5)	-	-	-	-	-	-
	Automatic	Eq. (6)	0.96	0	5.8	175.5	843	151
		Eq. (7)	0.91	0	11.9	173.0	373	43
Mixed	Constant	Eq. (5)	1.01	-	17.5	-	9064	-
	Automatic	Eq. (6)	1.01	0	24.5	172.6	714	43
		Eq. (7)	1.04	0.01	34.08	161.8	298	213

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Table 5. Results of numerical solution of example 1 using fully implicit scheme with tangent approximation ($t=360$ s and $\Delta z = 1$ cm)

	Time Stepping Method	Convergence criterion	MB		RMSE (cm)		Iteration Numbers	
			Δt (s)		Δt (s)		Δt (s)	
			10	30	10	30	10	30
h- based	Constant	Eq. (5)	0.92	0.87	0.06	0.27	173	104
		Eq. (6)	0.94	0.91	0.07	0.10	330	205
	Automatic	Eq. (7)	0.89	0.87	0.17	0.28	63	55
Mixed	Constant	Eq. (5)	1	1	0.03	0.18	172	100
		Eq. (6)	1	1	0.16	0.3	263	204
	Automatic	Eq. (7)	1.01	1.01	0.26	0.2	67	53

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Table 6. Results of numerical solution of example 2 using fully implicit scheme with tangent approximation ($t=86400$ s and $\Delta z = 2.5$ cm)

			MB		RMSE (cm)		Iteration Numbers	
Time Stepping Method		Convergence criterion	Δt (s)		Δt (s)		Δt (s)	
			20	144	20	144	20	144
h- based	Constant	Eq. (5)	0.98	-	15.8	-	9135	-
	Automatic	Eq. (6)	0.95	0.95	3.9	4.0	819	824
		Eq. (7)	0.87	0.88	29.9	23.3	322	327
Mixed	Constant	Eq. (5)	1	1	17.1	16.7	9052	1936
	Automatic	Eq. (6)	1	1	10.9	17.6	659	603
		Eq. (7)	1	1	17.3	24.9	253	225

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Table 7. Results of numerical solution of example 1 by fully implicit scheme and using constant time stepping, standard convergence criterion and SCS approximation ($\Delta z = 1\text{ cm}$, $t = 360\text{ s}$)

Δt (s)	MB			RMSE (cm)			Iteration Numbers		
	10	30	120	10	30	120	10	30	120
h-based	1	1	1	0.09	0.15	0.37	175	105	58
Mixed	1	1	1	0.09	0.15	0.37	175	105	58

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Table 8. Results of numerical solution of example 2 by fully implicit scheme and using constant time stepping, standard convergence criterion and SCS approximation ($\Delta z = 2.5 \text{ cm}$, $t = 86400 \text{ s}$)

Δt (s)	MB			RMSE (cm)			Iteration Numbers		
	20	144	720	20	144	720	20	144	720
h-based	1	1	1	16.9	16.7	16.7	9709	1977	757
Mixed	1	1	1	16.9	16.7	16.7	9709	1977	757

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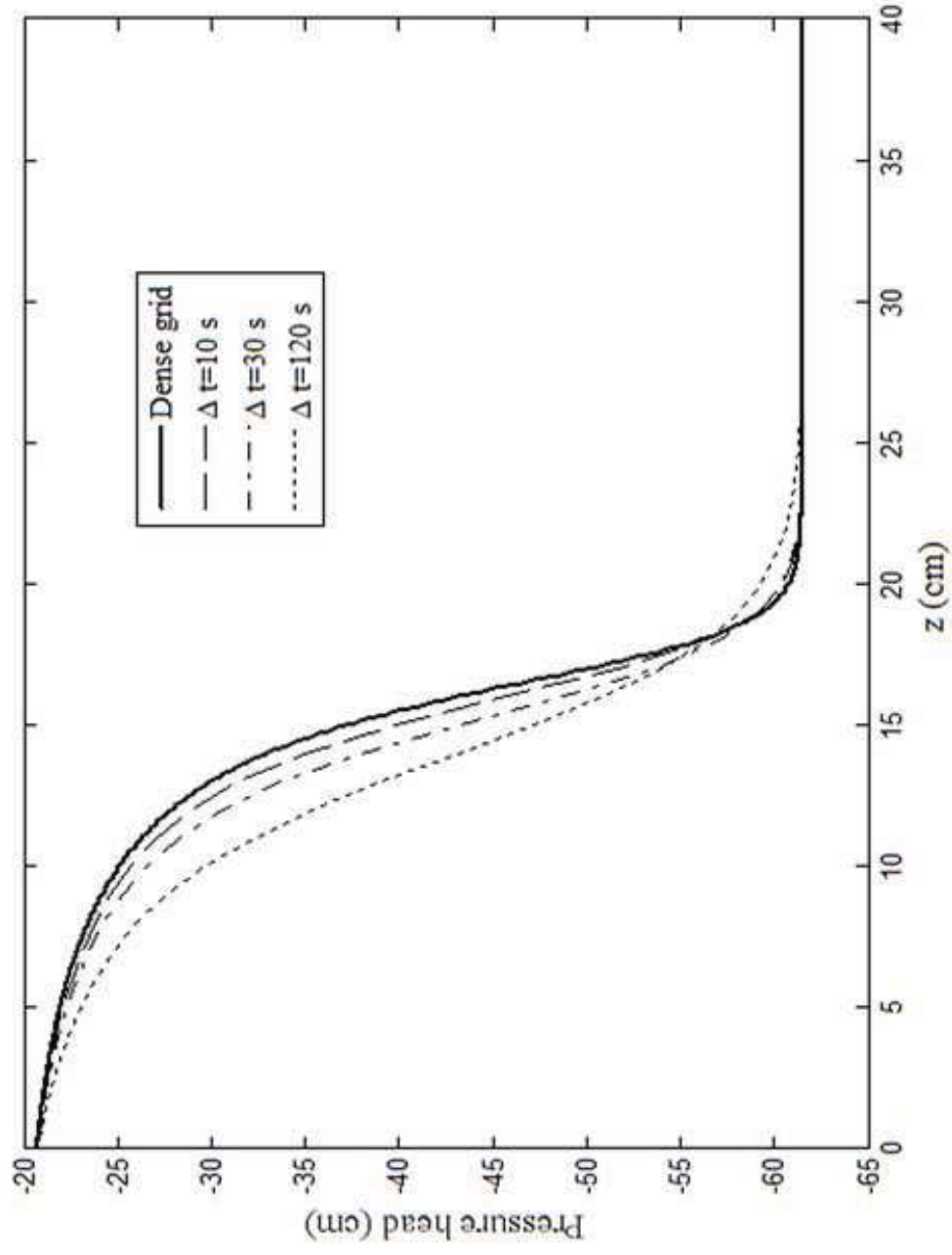
Figure Captions

Figure 1. Numerical solutions of example 1 using fully implicit scheme with tangent approximation for (a) h-based form and (b) Mixed-based form ($t=360$ s and $\Delta z=1$ cm).

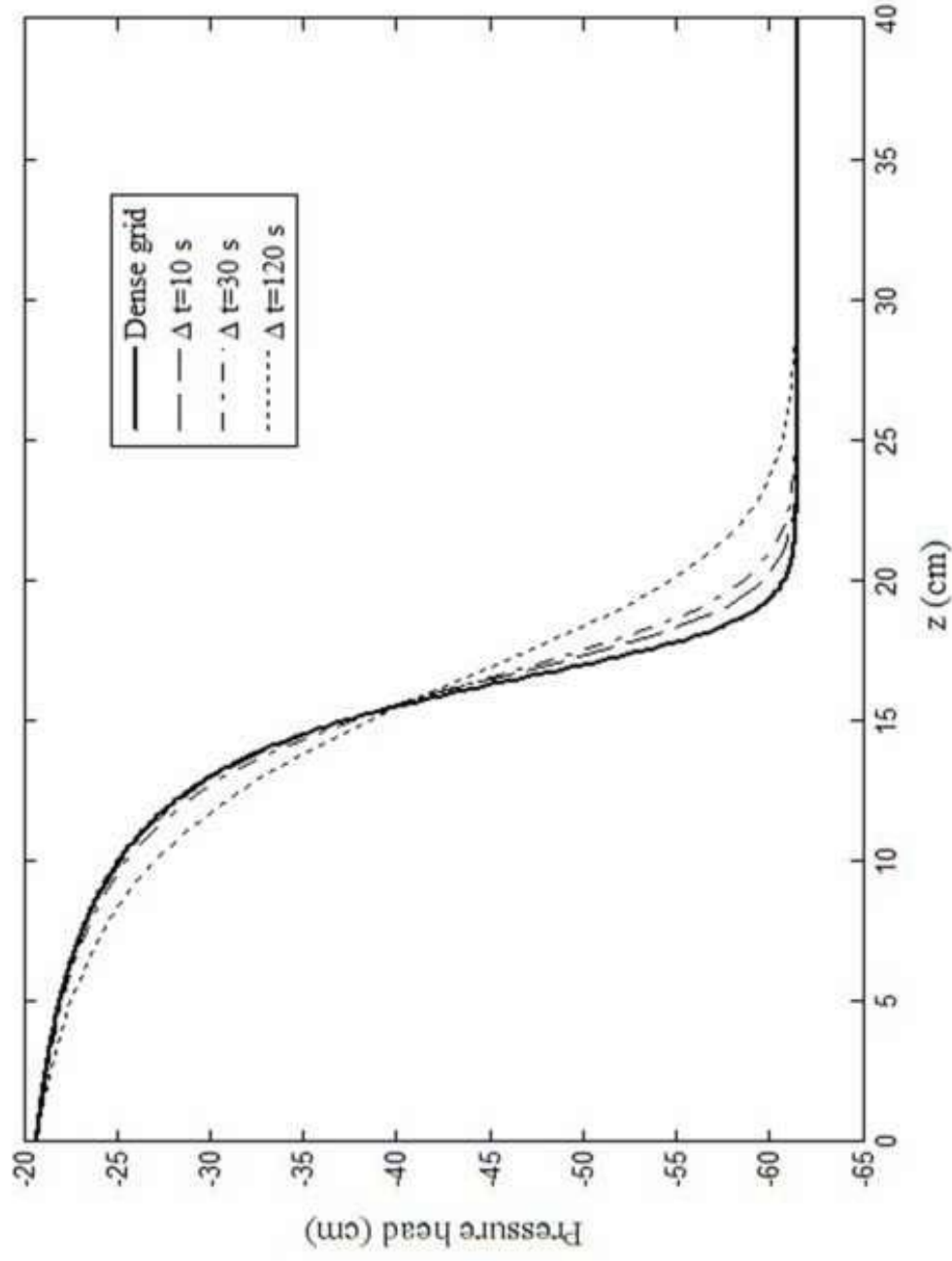
Figure 2. Numerical solutions of example 2 using fully implicit scheme with tangent approximation for (a) h-based form and (b) Mixed-based form ($t=86400$ s and $\Delta z=2.5$ cm).

Figure 3. Numerical solutions of example 1 using fully implicit scheme with SCS approximation for (a) h-based form ($t=360$ s and $\Delta z=1$ cm).

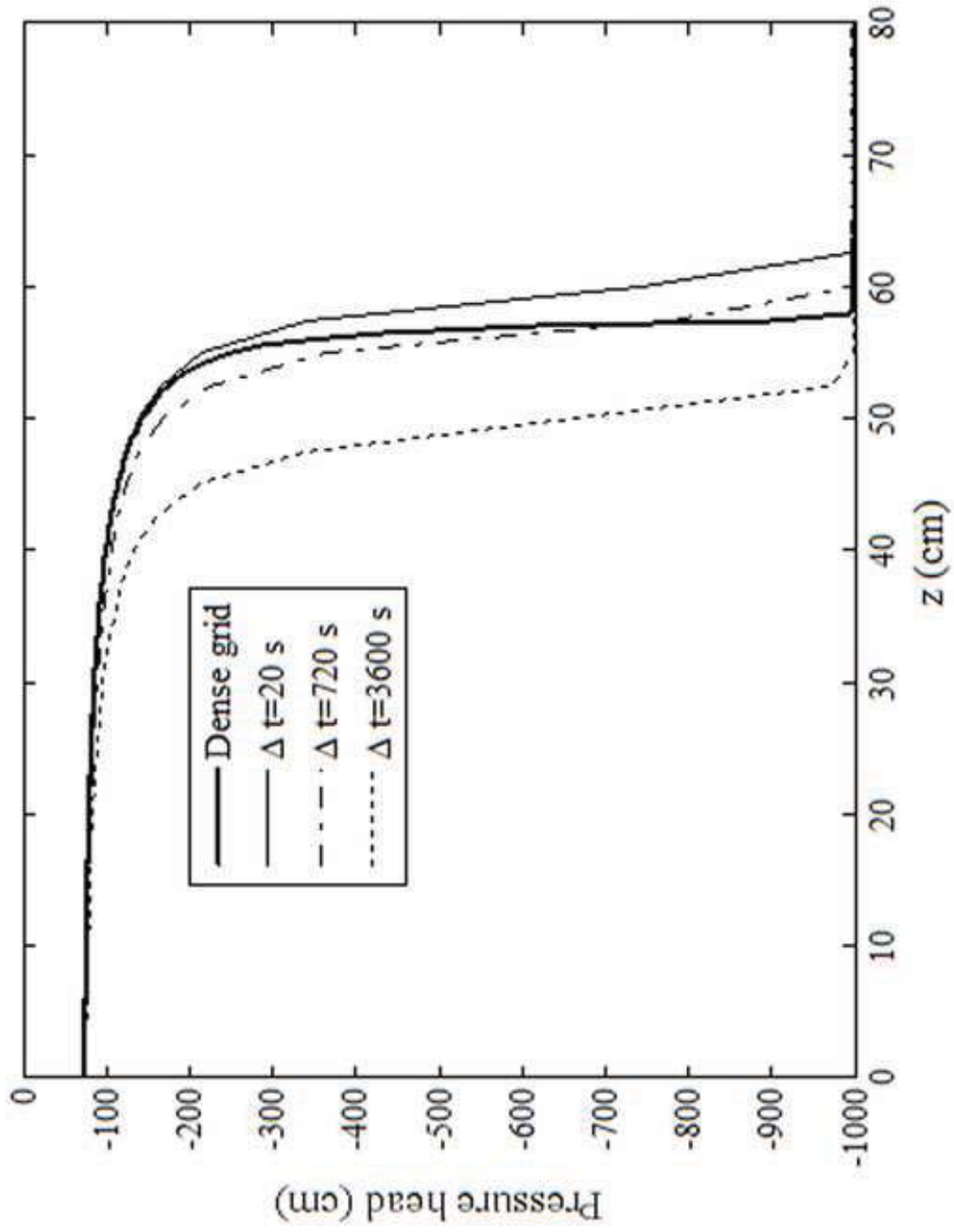
Figure 4. Numerical solutions of example 2 using fully implicit scheme with SCS approximation for h-based form ($t=86400$ s and $\Delta z=2.5$ cm).



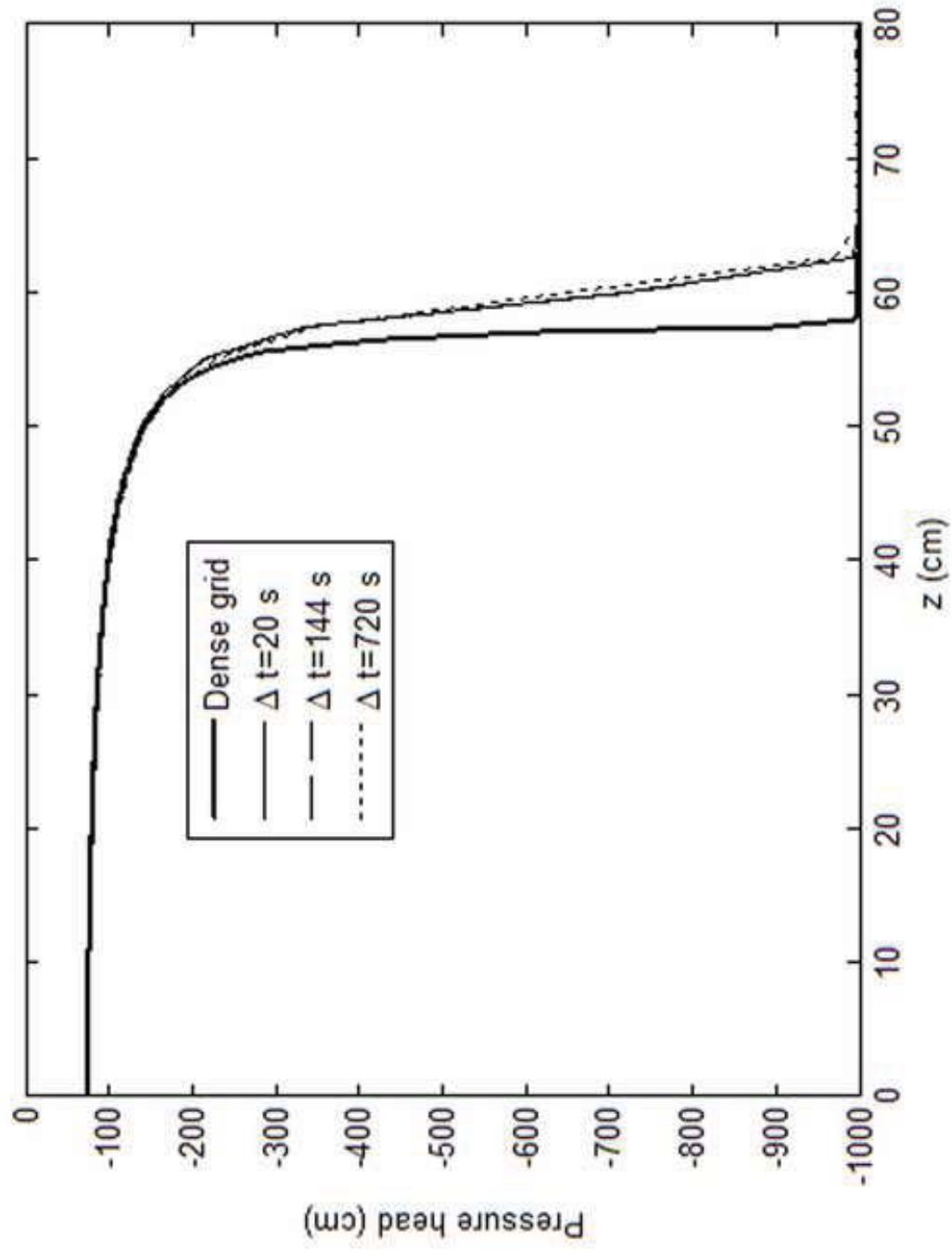
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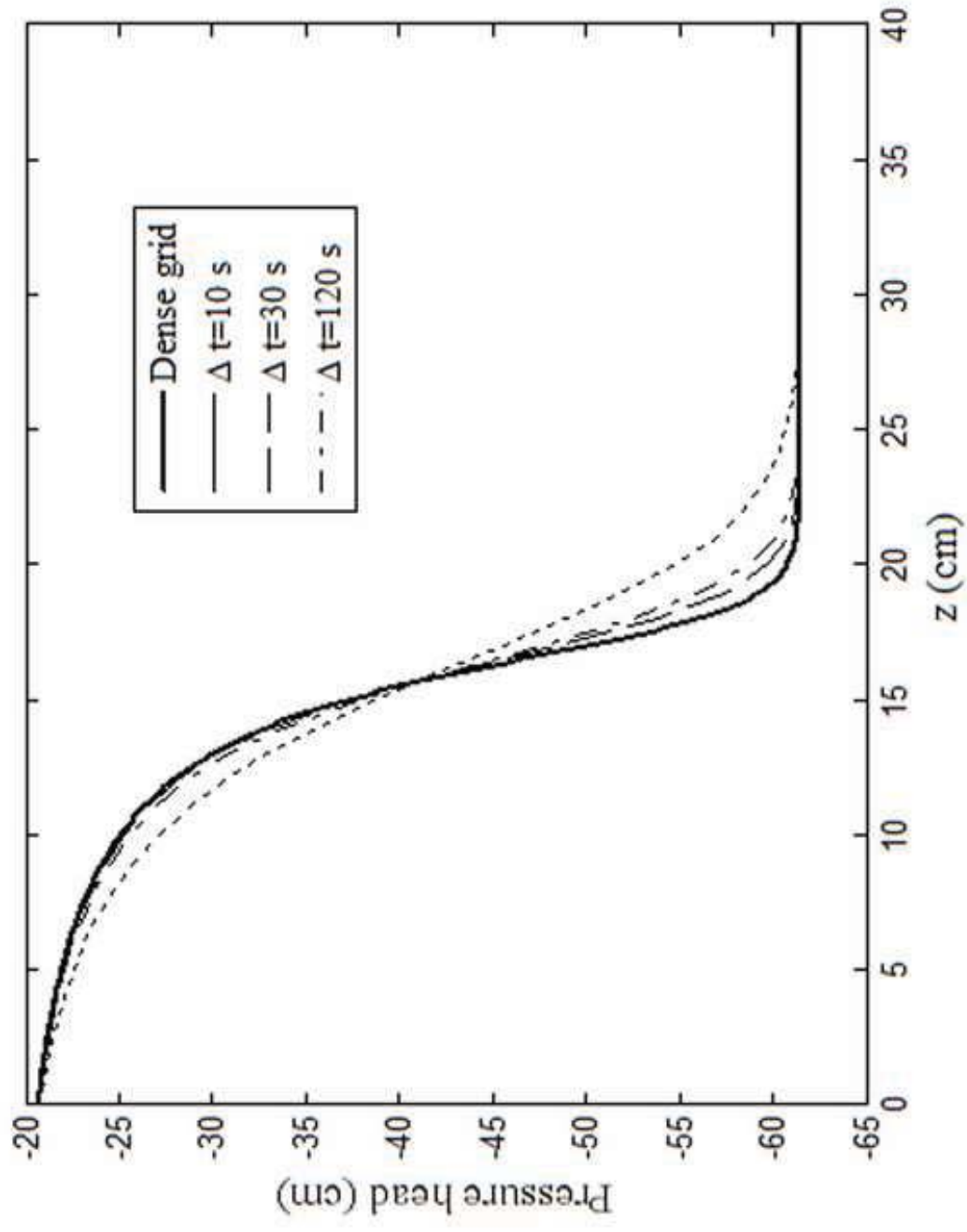
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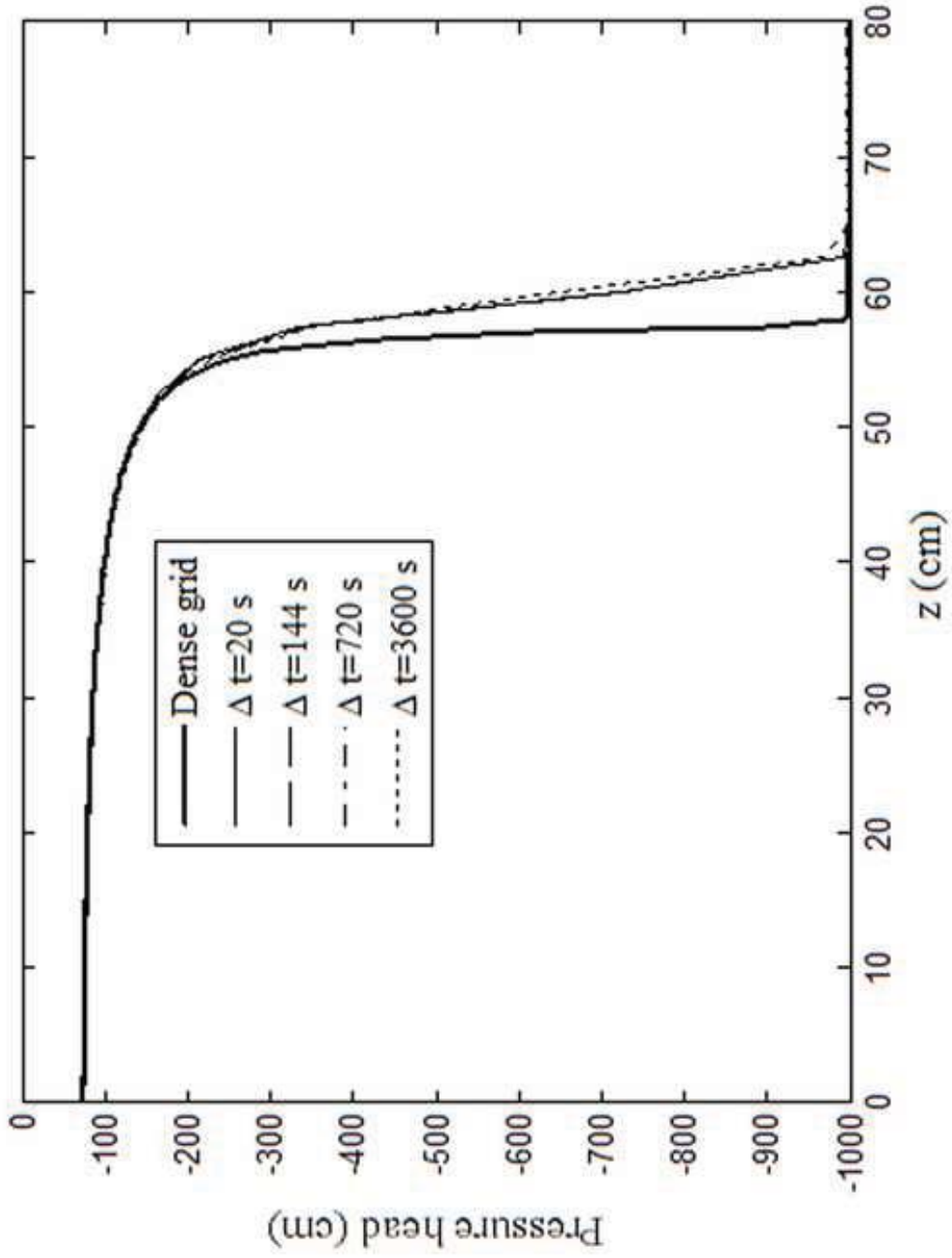
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